An Alternative Derivation of Johannisson's Regular Perturbation Model

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Abstract

We provide here an alternative derivation of the generalization of the nonlinear Turin model for dispersion unmanaged coherent optical links provided in Johannisson's report [1].

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I. INTRODUCTION

Goal of this paper is to provide a simplified derivation of the results appearing in the recent ArXiv posting of P. Johannisson [1] on a generalization of the well-known frequency-domain nonlinear interference (NLI) analytical model for dispersion unmanaged (DU) coherent systems introduced by Turin's group in [2].

II. FREQUENCY DOMAIN NLI RP1 SOLUTION

We start from the dual-polarization (DP) single-channel first-order Regular Perturbation (RP1) solution of the dispersion-managed nonlinear Schroedinger equation (DMNLSE) ([3], Appendix 2):

$$\tilde{\boldsymbol{U}}(L,f)=\tilde{\boldsymbol{U}}(0,f)+\tilde{\boldsymbol{U}}_p(L,f) \tag{1}$$

where L is the total link length, and the NLI *p*erturbation field is

$$\tilde{\boldsymbol{U}}_p(L,f)=-jP_0\iint_{-\infty}^{\infty}\mathcal{K}(f_1f_2)\tilde{\boldsymbol{U}}(0,f+f_1)\tilde{\boldsymbol{U}}^{\dagger}(0,f+f_1+f_2)\tilde{\boldsymbol{U}}(0,f+f_2)\underset{\cdot}{\mathrm{f}}_1\underset{\cdot}{\mathrm{f}}_2 \tag{2}$$

where:

i) boldface fields are 2x1 vectors containing the Fourier transforms (denoted by a tilde) of the X and Y polarizations in the transmitter polarization frame of reference; a dagger stands for transposition and conjugation; and the DP field power is normalized to an arbitrary reference power P_0 (which in [1] is chosen as the *per-polarization* average power);

ii) the un-normalized scalar frequency kernel is defined as (Cfr. [1] eq. (45) and [3] eq. (25)):

$$\mathcal{K}(F)\triangleq\int_0^L\gamma'(s)\mathcal{G}(s)\mathrm{e}^{-jC(s)(2\pi)^2F}ds \tag{3}$$

where $F=f_1f_2$ is the product of two frequencies, $\gamma'=\frac{8}{9}\gamma$ with γ the fiber nonlinear coefficient, $\mathcal{G}(s)$ the line power gain from $z=0$ to $z=s$, and $C(s)=-\int_0^s\beta_2(z)dz$ is the cumulated dispersion in the transmission fibers (with dispersion coefficient β_2) up to coordinate s. We choose here the frequency-normalizing rate in [3] as $R=1$, i.e., we do not normalize the frequency axis, as in [1]. In [3] we use the normalized kernel

$$\tilde{\eta}(F)=\frac{\mathcal{K}(F)}{\mathcal{K}(0)}$$

which then at $F=0$ equals 1. The nonlinear phase referred to power P_0 is

$$\Phi_{NL} = P_0\mathcal{K}(0).$$

Hence by multiplying and dividing by $\mathcal{K}(0)$ we can recast (2) as

$$\tilde{\boldsymbol{U}}_p(L,f) = -j\Phi_{NL}\iint_{-\infty}^{\infty}\tilde{\eta}(f_1f_2)\tilde{\boldsymbol{U}}(0,f+f_1)\tilde{\boldsymbol{U}}^{\dagger}(0,f+f_1+f_2)\tilde{\boldsymbol{U}}(0,f+f_2)\mathrm{f}_1\mathrm{f}_2 \tag{4}$$

From (4), the X component of the RP1 solution writes explicitly as

$$\begin{aligned}\frac{\tilde{U}_{x,p}(L,f)}{-j\Phi_{NL}} = &\iint_{-\infty}^{\infty}\tilde{\eta}(f_1f_2)\tilde{U}_x(0,f+f_1)\tilde{U}_x^*(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)\mathrm{f}_1\mathrm{f}_2+\\ &\iint_{-\infty}^{\infty}\tilde{\eta}(f_1f_2)\tilde{U}_x(0,f+f_1)\tilde{U}_y^*(0,f+f_1+f_2)\tilde{U}_y(0,f+f_2)\mathrm{f}_1\mathrm{f}_2\end{aligned} \tag{5}$$

where the first line gives the self-phase modulation (SPM) of X on X, while the second line gives the intra-channel cross-polarization modulation (I-XPolM) of Y on X. A perfectly dual expression for component Y is obtained by exchanging the indices x and y.

III. Gaussian Assumption and Johannisson's Result

In [1], [2] the key assumption is that the input fields are composed of independent spectral lines with *Gaussian* amplitude:

$$\tilde{U}_x(0,f) = \sqrt{f_0}\sum_{k=-\infty}^{\infty}\xi_k\sqrt{\hat{G}_x(kf_0)}\delta(f-kf_0)$$
$$\tilde{U}_y(0,f) = \sqrt{f_0}\sum_{k=-\infty}^{\infty}\zeta_k\sqrt{\hat{G}_y(kf_0)}\delta(f-kf_0)$$

with ξ_k and ζ_k independent identically distributed standard (i.e. zero-mean unit variance) circular complex Gaussian random variables (RV). Such signals do have a *per-polarization* power spectral density $\hat{G}_{x/y}(f)$ (normalized to P_0) in the limit $f_0 \to 0$ [2]. Then after long statistical averaging calculations, one gets the power spectral density of the $\tilde{U}_{x,p}(L,f)$ RV as ([1], eq. (89). Note that our PSD $\hat{G}(f)$ is normalized such that $G(f) \equiv P_0\hat{G}(f)$, where G is the un-normalized PSD per polarization. Also, $P_x = P_0\int_{-\infty}^{\infty}\hat{G}_x(f)df$ and $P_y = P_0\int_{-\infty}^{\infty}\hat{G}_y(f)df$)

$$\begin{aligned}\hat{G}_{x,p}(f) = &P_0^2\{2\iint_{-\infty}^{\infty}\left|\mathcal{K}((f_1-f)(f_2-f))\right|^2\hat{G}_x(f_1)\hat{G}_x(f_2)\hat{G}_x(f_1+f_2-f)\mathrm{f}_1\mathrm{f}_2\\ &+\iint_{-\infty}^{\infty}\left|\mathcal{K}((f_1-f)(f_2-f))\right|^2\hat{G}_x(f_1)\hat{G}_y(f_2)\hat{G}_y(f_1+f_2-f)\mathrm{f}_1\mathrm{f}_2\\ &+\mathcal{K}(0)^2\hat{G}_x(f)\left(4(\int_{-\infty}^{\infty}\hat{G}_x(f)df)^2+4\int_{-\infty}^{\infty}\hat{G}_x(f)df\int_{-\infty}^{\infty}\hat{G}_y(f)df+(\int_{-\infty}^{\infty}\hat{G}_y(f)df)^2\right)\}\end{aligned} \tag{6}$$

and a dual expression for Y is obtained by swapping $x \leftrightarrow y$. Recall that $\hat{G}_{x,p}(f)$ is the NLI PSD, normalized by P_0.

An equivalent form of (6) is the following

$$\begin{aligned}\hat{G}_{x,p}(f) = &\Phi_{NL}^2\{2\iint_{-\infty}^{\infty}\left|\tilde{\eta}(f_1f_2)\right|^2\hat{G}_x(f+f_1)\hat{G}_x(f+f_2)\hat{G}_x(f+f_1+f_2)\mathrm{f}_1\mathrm{f}_2\\ &+\iint_{-\infty}^{\infty}\left|\tilde{\eta}(f_1f_2)\right|^2\hat{G}_x(f+f_1)\hat{G}_y(f+f_2)\hat{G}_y(f+f_1+f_2)\mathrm{f}_1\mathrm{f}_2\\ &+\hat{G}_x(f)\left(4(\int_{-\infty}^{\infty}\hat{G}_x(f)df)^2+4\int_{-\infty}^{\infty}\hat{G}_x(f)df\int_{-\infty}^{\infty}\hat{G}_y(f)df+(\int_{-\infty}^{\infty}\hat{G}_y(f)df)^2\right)\}\end{aligned} \tag{7}$$

which better shows the formal parallel with the field equation (5): the field double integral in f_1, f_2 of the product kernel-field-field*-field becomes a PSD double integral in f_1, f_2 of the product squared kernel magnitude-PSD-PSD-PSD.

It is the purpose of the remaining part of this paper to provide a new proof of (7).

IV. The New Proof

We now start from (5) and make the following two assumptions regarding the input X,Y fields $U_x(0,t)$, $U_y(0,t)$:
1) they are wide-sense stationary (WSS);
2) they are jointly Gaussian processes.

Regarding assumption 1), we plan to exploit the following extension of result ([4], p. 418, eq. (12-76)):

***Theorem* 1**
Consider the jointly WSS stochastic processes $x(t)$ and $y(t)$, and let

$$\tilde{X}(f) \equiv \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-j2\pi ft} \mathrm{d}t$$

the Fourier transform of x (in the mean-square (MS) sense), and $\tilde{Y}(f)$ is similarly defined. Let their cross power spectral density (PSD) be $G_{xy}(f) = \mathcal{F}[R_{xy}(\tau)] = \mathcal{F}[E[x(t+\tau)y^*(t)]]$. Then

$$E[\tilde{X}(f)\tilde{Y}^*(u)] = G_{xy}(f)\delta(f-u) \equiv G_{xy}(f)\delta(u-f) \qquad \square \tag{8}$$

As a byproduct, we also have

$$E[\tilde{X}(f)\tilde{X}^*(u)] = G_x(f)\delta(f-u) \equiv G_x(f)\delta(u-f).$$

This theorem thus shows that the Fourier transform of *any MS-integrable WSS process* is nonstationary white noise, and thus *the spectral lines of its Fourier transform are uncorrelated*.

Regarding assumption 2), we plan to exploit the following result, known as the *complex Gaussian moment theorem* (CGMT), a generalization to complex variables of Isserlis theorem [5], [6]:

Theorem 2
Let $U_1, U_2, \ldots, U_{2k}$ be zero-mean jointly circular complex Gaussian random variables. Then

$$E[U_1^* U_2^* ... U_k^* U_{k+1} U_{k+2} ... U_{2k}] = \sum_\pi E[U_1^* U_p] E[U_2^* U_q] ..... E[U_k^* U_r] \tag{9}$$

where $\sum_\pi$ denotes a summation over the $k!$ possible permutations $(p, q, ..., r)$ of indices $(k+1, k+2, ..., 2k)$ $\square$

For instance,

$$\begin{aligned} E[U_1^* U_2^* U_3^* U_4 U_5 U_6] &= E[U_1^* U_4] E[U_2^* U_5] E[U_3^* U_6] \\ &+ E[U_1^* U_4] E[U_2^* U_6] E[U_3^* U_5] \\ &+ E[U_1^* U_5] E[U_2^* U_4] E[U_3^* U_6] \\ &+ E[U_1^* U_5] E[U_2^* U_6] E[U_3^* U_4] \\ &+ E[U_1^* U_6] E[U_2^* U_4] E[U_3^* U_5] \\ &+ E[U_1^* U_6] E[U_2^* U_5] E[U_3^* U_4]. \end{aligned} \tag{10}$$

Let's now start the new proof. We are interested in the PSD $\hat{G}_{x,p}(f)$ of the NLI field $U_{x,p}(L,t) = \mathcal{F}^{-1}[\tilde{U}_{x,p}(L,f)]$. By theorem 1 we have:

$$E[\tilde{U}_{x,p}(L,f)\tilde{U}^*_{x,p}(L,u)] = \hat{G}_{x,p}(f)\delta(u-f). \tag{11}$$

The left hand side can be explicitly calculated using (5):

$$\frac{E[\tilde{U}_{x,p}(L,f)\tilde{U}^*_{x,p}(L,u)]}{\Phi^2_{NL}} =$$

$$E[\iint_{-\infty}^{\infty} \tilde{\eta}(f_1 f_2)[\tilde{U}_x(0,f+f_1)\tilde{U}^*_x(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)+$$
$$\tilde{U}_x(0,f+f_1)\tilde{U}^*_y(0,f+f_1+f_2)\tilde{U}_y(0,f+f_2)]df_1 df_2\cdot$$
$$\iint_{-\infty}^{\infty} \tilde{\eta}(f_3 f_4)^*[\tilde{U}^*_x(0,u+f_3)\tilde{U}_x(0,u+f_3+f_4)\tilde{U}^*_x(0,u+f_4)+$$
$$\tilde{U}^*_x(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}^*_y(0,u+f_4)]df_3 df_4] =$$
$$\iiiint_{-\infty}^{\infty} df_1 df_2 df_3 df_4 \tilde{\eta}(f_1 f_2)\tilde{\eta}(f_3 f_4)^*\cdot$$
$$\{E[\tilde{U}_x(0,f+f_1)\tilde{U}^*_x(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)\tilde{U}^*_x(0,u+f_3)\tilde{U}_x(0,u+f_3+f_4)\tilde{U}^*_x(0,u+f_4)]+$$
$$E[\tilde{U}_x(0,f+f_1)\tilde{U}^*_y(0,f+f_1+f_2)\tilde{U}_y(0,f+f_2)\tilde{U}^*_x(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}^*_y(0,u+f_4)]+$$
$$2\mathrm{Re}(E[\tilde{U}_x(0,f+f_1)\tilde{U}^*_x(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)\tilde{U}^*_x(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}^*_y(0,u+f_4)])\}. \quad (12)$$

Now, putting together Theorems 1 and 2, Appendix 1 shows the following

Theorem 3

For jointly stationary circular complex Gaussian zero-mean processes $A(t), B(t), C(t), D(t), E(t), F(t)$ we have the general formula

$$E\left[\tilde{A}(f+f_1)\tilde{B}^*(f+f_1+f_2)\tilde{C}(f+f_2)\tilde{D}^*(u+f_3)\tilde{E}(u+f_3+f_4)\tilde{F}^*(u+f_4)\right] =$$
$$\Big[G_{ab}(f+f_1)G_{cd}(f)G_{ef}(f+f_4)\delta(f_2)\delta(f_3)+$$
$$G_{ab}(f+f_1)G_{ed}(f+f_3)G_{cf}(f)\delta(f_2)\delta(f_4)+$$
$$G_{cb}(f+f_2)G_{ad}(f)G_{ef}(f+f_4)\delta(f_1)\delta(f_3)+$$
$$G_{cb}(f+f_2)G_{ed}(f+f_3)G_{af}(f)\delta(f_1)\delta(f_4)+$$
$$G_{eb}(f+f_1+f_2)G_{ad}(f+f_1)G_{cf}(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)+$$
$$G_{eb}(f+f_1+f_2)G_{cd}(f+f_2)G_{af}(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\Big]$$
$$\cdot\delta(u-f) \qquad \square \qquad (13)$$

We next apply the general formula (13) to the three expectations in (12) to get:

First expectation:

$$E\left[\tilde{U}_x(0,f+f_1)\tilde{U}^*_x(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)\tilde{U}^*_x(0,u+f_3)\tilde{U}_x(0,u+f_3+f_4)\tilde{U}^*_x(0,u+f_4)\right] =$$
$$\delta(u-f)\Big[G_{xx}(f+f_1)G_{xx}(f)G_{xx}(f+f_4)\delta(f_2)\delta(f_3)+$$
$$G_{xx}(f+f_1)G_{xx}(f+f_3)G_{xx}(f)\delta(f_2)\delta(f_4)+$$
$$G_{xx}(f+f_2)G_{xx}(f)G_{xx}(f+f_4)\delta(f_1)\delta(f_3)+$$
$$G_{xx}(f+f_2)G_{xx}(f+f_3)G_{xx}(f)\delta(f_1)\delta(f_4)+$$
$$G_{xx}(f+f_1+f_2)G_{xx}(f+f_1)G_{xx}(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)+$$
$$G_{xx}(f+f_1+f_2)G_{xx}(f+f_2)G_{xx}(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\Big] \quad (14)$$

where $G_{xx} \equiv \hat{G}_x$. Second expectation:

$$E\left[\tilde{U}_x(0,f+f_1)\tilde{U}_y^*(0,f+f_1+f_2)\tilde{U}_y(0,f+f_2)\tilde{U}_x^*(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}_y^*(0,u+f_4)\right] =$$
$$\delta(u-f)\Big[G_{xy}(f+f_1)G_{yx}(f)G_{yy}(f+f_4)\delta(f_2)\delta(f_3)+$$
$$G_{xy}(f+f_1)G_{yx}(f+f_3)G_{yy}(f)\delta(f_2)\delta(f_4)+$$
$$G_{yy}(f+f_2)G_{xx}(f)G_{yy}(f+f_4)\delta(f_1)\delta(f_3)+$$
$$G_{yy}(f+f_2)G_{yx}(f+f_3)G_{xy}(f)\delta(f_1)\delta(f_4)+$$
$$G_{yy}(f+f_1+f_2)G_{xx}(f+f_1)G_{yy}(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)+$$
$$G_{yy}(f+f_1+f_2)G_{yx}(f+f_2)G_{xy}(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\Big] \tag{15}$$

where $G_{yy} \equiv \hat{G}_y$, and assuming uncorrelated X and Y we get

$$E\left[\tilde{U}_x(0,f+f_1)\tilde{U}_y^*(0,f+f_1+f_2)\tilde{U}_y(0,f+f_2)\tilde{U}_x^*(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}_y^*(0,u+f_4)\right] =$$
$$\delta(u-f)\Big[G_{yy}(f+f_2)G_{xx}(f)G_{yy}(f+f_4)\delta(f_1)\delta(f_3)+$$
$$G_{yy}(f+f_1+f_2)G_{xx}(f+f_1)G_{yy}(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)\Big]. \tag{16}$$

Third expectation:

$$E\left[\tilde{U}_x(0,f+f_1)\tilde{U}_x^*(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)\tilde{U}_x^*(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}_y^*(0,u+f_4)\right] =$$
$$\delta(u-f)\Big[G_{xx}(f+f_1)G_{xx}(f)G_{yy}(f+f_4)\delta(f_2)\delta(f_3)+$$
$$G_{xx}(f+f_1)G_{yx}(f+f_3)G_{xy}(f)\delta(f_2)\delta(f_4)+$$
$$G_{xx}(f+f_2)G_{xx}(f)G_{yy}(f+f_4)\delta(f_1)\delta(f_3)+$$
$$G_{xx}(f+f_2)G_{yx}(f+f_3)G_{xy}(f)\delta(f_1)\delta(f_4)+$$
$$G_{yx}(f+f_1+f_2)G_{xx}(f+f_1)G_{xy}(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)+$$
$$G_{yx}(f+f_1+f_2)G_{xx}(f+f_2)G_{xy}(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\Big] \tag{17}$$

and assuming uncorrelated X and Y we get

$$E\left[\tilde{U}_x(0,f+f_1)\tilde{U}_x^*(0,f+f_1+f_2)\tilde{U}_x(0,f+f_2)\tilde{U}_x^*(0,u+f_3)\tilde{U}_y(0,u+f_3+f_4)\tilde{U}_y^*(0,u+f_4)\right] =$$
$$\delta(u-f)\Big[G_{xx}(f+f_1)G_{xx}(f)G_{yy}(f+f_4)\delta(f_2)\delta(f_3)+$$
$$G_{xx}(f+f_2)G_{xx}(f)G_{yy}(f+f_4)\delta(f_1)\delta(f_3)\Big]. \tag{18}$$

Substitution of (14),(16),(18) into (12) finally gives

$$\frac{E[\tilde{U}_{x,p}(L,f)\tilde{U}_{x,p}^*(L,u)]}{\Phi_{NL}^2} = \delta(u-f)\{\iiiint_{-\infty}^{\infty} \mathrm{f}_1\mathrm{f}_2\mathrm{f}_3\mathrm{f}_4\tilde{\eta}(f_1f_2)\tilde{\eta}(f_3f_4)^*\cdot$$
$$\cdot\{\Big[\hat{G}_x(f+f_1)\hat{G}_x(f)\hat{G}_x(f+f_4)\delta(f_2)\delta(f_3)+\hat{G}_x(f+f_1)\hat{G}_x(f+f_3)\hat{G}_x(f)\delta(f_2)\delta(f_4)+$$
$$\hat{G}_x(f+f_2)\hat{G}_x(f)\hat{G}_x(f+f_4)\delta(f_1)\delta(f_3)+\hat{G}_x(f+f_2)\hat{G}_x(f+f_3)\hat{G}_x(f)\delta(f_1)\delta(f_4)+$$
$$\hat{G}_x(f+f_1+f_2)\hat{G}_x(f+f_1)\hat{G}_x(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)+$$
$$\hat{G}_x(f+f_1+f_2)\hat{G}_x(f+f_2)\hat{G}_x(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\Big]+$$
$$\Big[\hat{G}_y(f+f_2)\hat{G}_x(f)\hat{G}_y(f+f_4)\delta(f_1)\delta(f_3)+\hat{G}_y(f+f_1+f_2)\hat{G}_x(f+f_1)\hat{G}_y(f+f_2)\delta(f_3-f_1)\delta(f_4-f_2)\Big]+$$
$$2\mathrm{Re}(\Big[\hat{G}_x(f+f_1)\hat{G}_x(f)\hat{G}_y(f+f_4)\delta(f_2)\delta(f_3)+\hat{G}_x(f+f_2)\hat{G}_x(f)\hat{G}_y(f+f_4)\delta(f_1)\delta(f_3)\Big])\}\}.$$

From (11), the term multiplying $\delta(u-f)$ must be the desired PSD. Each pair of delta removes two integrals, so that the PSD turns out to be (first two lines above produce first 4 lines, third line above produces 5th line, 4th line above produces 6th and 7th lines, and final line above produces the last two lines):

$$
\begin{aligned}
\frac{\hat{G}_{x,p}(f)}{\Phi_{NL}^2} &= |\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_x(f+f_1)\hat{G}_x(f)\hat{G}_x(f+f_4)\mathrm{f}_1\mathrm{f}_4 \\
&+ |\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_x(f+f_1)\hat{G}_x(f+f_3)\hat{G}_x(f)\mathrm{f}_1\mathrm{f}_3 \\
&+ |\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_x(f+f_2)\hat{G}_x(f)\hat{G}_x(f+f_4)\mathrm{f}_2\mathrm{f}_4 \\
&+ |\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_x(f+f_2)\hat{G}_x(f+f_3)\hat{G}_x(f)\mathrm{f}_2\mathrm{f}_3 \\
&+ 2\iint_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \hat{G}_x(f+f_1+f_2)\hat{G}_x(f+f_1)\hat{G}_x(f+f_2)\mathrm{f}_1\mathrm{f}_2 \\
&+ |\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_y(f+f_2)\hat{G}_x(f)\hat{G}_y(f+f_4)\mathrm{f}_2\mathrm{f}_4 \\
&+ \iint_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \hat{G}_y(f+f_1+f_2)\hat{G}_x(f+f_1)\hat{G}_y(f+f_2)\mathrm{f}_1\mathrm{f}_2 \\
&+ 2\Bigg(|\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_x(f+f_1)\hat{G}_x(f)\hat{G}_y(f+f_4)\mathrm{f}_1\mathrm{f}_4 + \\
&\quad |\tilde{\eta}(0)|^2 \iint_{-\infty}^{\infty} \hat{G}_x(f+f_2)\hat{G}_x(f)\hat{G}_y(f+f_4)\mathrm{f}_2\mathrm{f}_4 \Bigg).
\end{aligned}
$$

In summary, considering that by construction $\tilde{\eta}(0)=1$, we have:

$$
\begin{aligned}
\frac{\hat{G}_{x,p}(f)}{\Phi_{NL}^2} &= 2\iint_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \hat{G}_x(f+f_1+f_2)\hat{G}_x(f+f_1)\hat{G}_x(f+f_2)\mathrm{f}_1\mathrm{f}_2 \\
&+ \iint_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \hat{G}_y(f+f_1+f_2)\hat{G}_x(f+f_1)\hat{G}_y(f+f_2)\mathrm{f}_1\mathrm{f}_2 \\
&+ \hat{G}_x(f)\left(4\Big(\int_{-\infty}^{\infty} \hat{G}_x(f)df\Big)^2 + 4\int_{-\infty}^{\infty} \hat{G}_x(f)df \int_{-\infty}^{\infty} \hat{G}_y(f)df + 1\Big(\int_{-\infty}^{\infty} \hat{G}_y(f)df\Big)^2 \right)
\end{aligned}
$$

which confirms Johannisson's equation (7) and completes the desired alternative proof.

V. Conclusions

We have presented an alternative derivation of Johannisson's result [1]. We first remark that our new method is able to deal with correlated X and Y, although this feature was not exploited in the present paper. Next we note that we did not have to assume independent input spectral lines: this comes naturally from the stationarity of the input process. Finally, the truly critical assumption in the model in [1], [2] is therefore the assumption of Gaussianity at any z during propagation, which is implicit in the assumption of a Gaussian input process, and the fact that the "forcing terms" in the RP equation are the linearly distorted signals at any z, which thus remain Gaussian.

Therefore the true limit of the model in [1], [2] is that indeed starting from a non-Gaussian spectrum such as the one of a digitally modulated signal[1], it takes some finite propagation in a non-infinite dispersion line to approximately get both a Gaussian spectrum and a Gaussian-like time-domain signal.

APPENDIX 1

In this Appendix we prove Theorem 3 in the text. Assuming jointly stationary circular complex Gaussian zero-mean processes $A(t), B(t), C(t), D(t), E(t), F(t)$, we have by using (8) in (10):

$$
\begin{aligned}
T &\triangleq E\left[\tilde{A}(f+f_1)\tilde{B}^*(f+f_1+f_2)\tilde{C}(f+f_2)\tilde{D}^*(u+f_3)\tilde{E}(u+f_3+f_4)\tilde{F}^*(u+f_4)\right] = \\
&\underbrace{E[\tilde{B}^*(f+f_1+f_2)\tilde{A}(f+f_1)]}_{G_{ab}(f+f_1)\delta(f_2)}\underbrace{E[\tilde{D}^*(u+f_3)\tilde{C}(f+f_2)]}_{G_{cd}(f+f_2)\delta(u+f_3-f-f_2)}\underbrace{E[\tilde{F}^*(u+f_4)\tilde{E}(u+f_3+f_4)]}_{G_{ef}(u+f_3+f_4)\delta(f_3)}+ \\
&\underbrace{E[\tilde{B}^*(f+f_1+f_2)\tilde{A}(f+f_1)]}_{G_{ab}(f+f_1)\delta(f_2)}\underbrace{E[\tilde{D}^*(u+f_3)\tilde{E}(u+f_3+f_4)]}_{G_{ed}(u+f_3+f_4)\delta(f_4)}\underbrace{E[\tilde{F}^*(u+f_4)\tilde{C}(f+f_2)]}_{G_{cf}(f+f_2)\delta(u+f_4-f-f_2)}+ \\
&\underbrace{E[\tilde{B}^*(f+f_1+f_2)\tilde{C}(f+f_2)]}_{G_{cb}(f+f_2)\delta(f_1)}\underbrace{E[\tilde{D}^*(u+f_3)\tilde{A}(f+f_1)]}_{G_{ad}(f+f_1)\delta(u+f_3-f-f_1)}\underbrace{E[\tilde{F}^*(u+f_4)\tilde{E}(u+f_3+f_4)]}_{G_{ef}(u+f_3+f_4)\delta(f_3)}+ \\
&\underbrace{E[\tilde{B}^*(f+f_1+f_2)\tilde{C}(f+f_2)]}_{G_{cb}(f+f_2)\delta(f_1)}\underbrace{E[\tilde{D}^*(u+f_3)\tilde{E}(u+f_3+f_4)]}_{G_{ed}(u+f_3+f_4)\delta(f_4)}\underbrace{E[\tilde{F}^*(u+f_4)\tilde{A}(f+f_1)]}_{G_{af}(f+f_1)\delta(u+f_4-f-f_1)}+ \\
&\underbrace{E[\tilde{B}^*(f+f_1+f_2)\tilde{E}(u+f_3+f_4)]}_{G_{eb}(u+f_3+f_4)\delta(f+f_1+f_2-u-f_3-f_4)}\underbrace{E[\tilde{D}^*(u+f_3)\tilde{A}(f+f_1)]}_{G_{ad}(f+f_1)\delta(u+f_3-f-f_1)}\underbrace{E[\tilde{F}^*(u+f_4)\tilde{C}(f+f_2)]}_{G_{cf}(f+f_2)\delta(u+f_4-f-f_2)}+ \\
&\underbrace{E[\tilde{B}^*(f+f_1+f_2)\tilde{E}(u+f_3+f_4)]}_{G_{eb}(u+f_3+f_4)\delta(f+f_1+f_2-u-f_3-f_4)}\underbrace{E[\tilde{D}^*(u+f_3)\tilde{C}(f+f_2)]}_{G_{cd}(f+f_2)\delta(u+f_3-f-f_2)}\underbrace{E[\tilde{F}^*(u+f_4)\tilde{A}(f+f_1)]}_{G_{af}(f+f_1)\delta(u+f_4-f-f_1)} =
\end{aligned}
$$

thus

$$
\begin{aligned}
T = {} & G_{ab}(f+f_1)G_{cd}(f+f_2)G_{ef}(u+f_3+f_4)\delta(f_2)\delta(f_3)\delta(u+f_3-f-f_2) \\
&+ G_{ab}(f+f_1)G_{ed}(u+f_3+f_4)G_{cf}(f+f_2)\delta(f_2)\delta(f_4)\delta(u+f_4-f-f_2) \\
&+ G_{cb}(f+f_2)G_{ad}(f+f_1)G_{ef}(u+f_3+f_4)\delta(f_1)\delta(f_3)\delta(u+f_3-f-f_1) \\
&+ G_{cb}(f+f_2)G_{ed}(u+f_3+f_4)G_{af}(f+f_1)\delta(f_1)\delta(f_4)\delta(u+f_4-f-f_1) \\
&+ G_{eb}(u+f_3+f_4)G_{ad}(f+f_1)G_{cf}(f+f_2)\cdot \\
&\cdot\delta(u+f_4-f-f_2)\delta(u+f_3-f-f_1)\delta(f+f_1+f_2-u-f_3-f_4) \\
&+ G_{eb}(u+f_3+f_4)G_{cd}(f+f_2)G_{af}(f+f_1)\cdot \\
&\cdot\delta(u+f_4-f-f_1)\delta(u+f_3-f-f_2)\delta(f+f_1+f_2-u-f_3-f_4).
\end{aligned}
$$

Now we use the sampling property of the delta to write, e.g. for the first line where $f_2=0$ and $f_3=0$,

$$G_{ab}(f+f_1)G_{cd}(f)G_{ef}(u+f_4)\delta(f_2)\delta(f_3)\delta(u-f)$$

and e.g. for the last line where $u+f_4=f+f_1$ and $u+f_3=f+f_2$ which we add up to get

$$u+f_3+f_4=(f-u)+f+f_1+f_2$$

whence

$$f+f_1+f_2-u-f_3-f_4=u-f$$

so that the last line writes as

$$
\begin{aligned}
&G_{eb}(u+f_3+f_4)G_{cd}(f+f_2)G_{af}(f+f_1)\delta(u+f_4-f-f_1)\delta(u+f_3-f-f_2)\delta(f+f_1+f_2-u-f_3-f_4) = \\
&G_{eb}((f-u)+f+f_1+f_2)G_{cd}(f+f_2)G_{af}(f+f_1)\cdot\delta(u+f_4-f-f_1)\delta(u+f_3-f-f_2)\delta(u-f) \overset{(\text{use } u=f)}{=} \\
&G_{eb}(f+f_1+f_2)G_{cd}(f+f_2)G_{af}(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\delta(u-f).
\end{aligned}
$$

[1]Although the authors in [2] present in their Appendix B an appealing heuristic justification of their Gaussian signal assumption, still their invoking the central limit theorem at their equation (37) is *not rigorous*. They would conclude that any digitally modulated signal with any number of levels has a Gaussian Fourier transform (which in turn implies the time-domain signal itself is Gaussian), which is clearly *not* the case.

We therefore get

$$
\begin{aligned}
T = {} & G_{ab}(f+f_1)G_{cd}(f)G_{ef}(f+f_4)\delta(f_2)\delta(f_3)\delta(u-f) \\
& + G_{ab}(f+f_1)G_{ed}(f+f_3)G_{cf}(f)\delta(f_2)\delta(f_4)\delta(u-f) \\
& + G_{cb}(f+f_2)G_{ad}(f)G_{ef}(f+f_4)\delta(f_1)\delta(f_3)\delta(u-f) \\
& + G_{cb}(f+f_2)G_{ed}(f+f_3)G_{af}(f)\delta(f_1)\delta(f_4)\delta(u-f) \\
& + G_{eb}(f+f_1+f_2)G_{ad}(f+f_1)G_{cf}(f+f_2)\delta(f_4-f_2)\delta(f_3-f_1)\delta(u-f) \\
& + G_{eb}(f+f_1+f_2)G_{cd}(f+f_2)G_{af}(f+f_1)\delta(f_4-f_1)\delta(f_3-f_2)\delta(u-f).
\end{aligned}
$$

whence the final form (13) given in Theorem 3.